14/9/23

MATH4030 Lecture

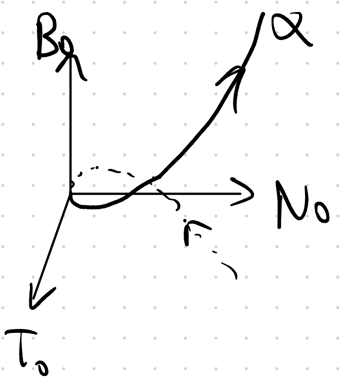
Recap:

- Locally at $s_0 \in I$, $\alpha(s) = \left(s - \frac{1}{6}k_0^2 s^3,\ \frac{1}{2}k_0 s^2 + \frac{1}{6}k_0' s^3,\ -\frac{1}{6}k_0 \tau_0 s^3\right) + o(s^3)$

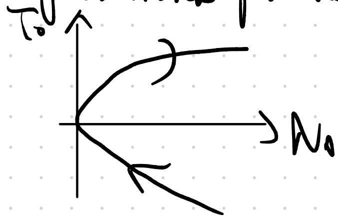
  in coordinates induced by $\{T_0, N_0, B_0\}$.

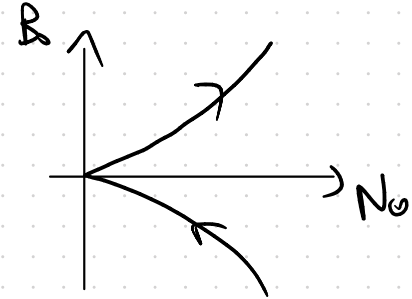
  Local Canonical Form ↑ (according to do Carmo).

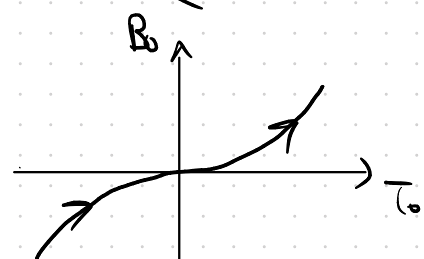
- Illustration:



Projection onto span$\{T_0, N_0\}$.







Existence Part of Fundamental Thm of Local Theory of Curves

Thm from ODE: Given initial condition $s_0 \in I$, $(z_1)_0, \dots, (z_q)_0$, there exists an open interval $J \subset I$ with $s_0 \in J$ and a unique differential mapping $\alpha : J \to \mathbb{R}^q$ with

$$\alpha(s_0) = ((z_1)_0, \dots, (z_q)_0)$$

$$\alpha'(s) = (f_1, \dots, f_q)$$

where f_i, $i = 1, \dots, q$ are functions of $(s, \alpha(s)) \in J \times \mathbb{R}^q$.

Furthermore, if the system is linear, then we can take $J = I$.

(Reference: Serge Lang, Undergraduate Analysis, 18.3 "Linear Differential Equations" Thm 3.1)

Pf of Existence: Recall Frenet's formulas:

$$\begin{bmatrix} T \\ N \\ B \end{bmatrix}' = \begin{bmatrix} 0 & k & 0 \\ -k & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}, \quad s \in I.$$

can be considered a differentiable system in $I \times \mathbb{R}^9$ (each vector $T, N, B \in \mathbb{R}^3$).

i.e.

$$\begin{cases} \dfrac{dz_1}{ds} = f_1(s, z_1, \ldots, z_9) \\ \quad \vdots \\ \dfrac{dz_9}{ds} = f_9(s, z_1, \ldots, z_9) \end{cases}, \quad s \in I$$

where $(z_1, z_2, z_3) = T$, $(z_4, z_5, z_6) = N$, $(z_7, z_8, z_9) = B$.

f_i, $i = 1, \ldots, 9$ are linear functions (w/ coefficients that may depend on s) of z_i.

Explicitly,

$T, N, B \in \mathbb{R}^3$.

$T = (T_1, T_2, T_3)$ $\quad$ $T_i, N_i, B_i \in \mathbb{R}$

$N = (N_1, N_2, N_3)$

$B = (B_1, B_2, B_3)$.

$$\begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ N_1 \\ N_2 \\ N_3 \\ B_1 \\ B_2 \\ B_3 \end{bmatrix}' = \begin{bmatrix} 0 & \begin{matrix} k & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & k \end{matrix} & 0 \\ \begin{matrix} -k & 0 & 0 \\ 0 & -k & 0 \\ 0 & 0 & -k \end{matrix} & 0 & \begin{matrix} \tau & 0 & 0 \\ 0 & \tau & 0 \\ 0 & 0 & \tau \end{matrix} \\ 0 & \begin{matrix} -\tau & 0 & 0 \\ 0 & -\tau & 0 \\ 0 & 0 & -\tau \end{matrix} & 0 \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ N_1 \\ N_2 \\ N_3 \\ B_1 \\ B_2 \\ B_3 \end{bmatrix}$$

$T_1' = kN_1$

$T_2' = kN_2 \Rightarrow T' = (T_1', T_2', T_3') = (kN_1, kN_2, kN_3) = kN$.

$T_3' = kN_3$

and so on.

Then by ODE thm, given initial data $s_0 \in I$

$$T_0 = ((z_1)_0, (z_2)_0, (z_3)_0), \quad N_0 = ((z_4)_0, (z_5)_0, (z_6)_0)$$

$$B_0 = ((z_7)_0, (z_8)_0, (z_9)_0)$$

there exists a family of frames $\{T(s), N(s), B(s)\}$, $s \in J \overset{\text{linear}}{=} I$,

s.t. $\begin{bmatrix} T \\ N \\ B \end{bmatrix}'$ satisfies Frenet formulas.

We need to check that $\{T(s), N(s), B(s)\}$ remain orthonormal for $s \in I$.

$$\frac{d}{ds}\langle T, N\rangle = \langle T', N\rangle + \langle T, N'\rangle$$

$$= k\langle N, N\rangle - k\langle T, T\rangle + \tau\langle T, B\rangle$$

$$\frac{d}{ds}\langle T, B\rangle = k\langle N, B\rangle - \tau\langle T, N\rangle$$

$$\frac{d}{ds}\langle N, B\rangle = -k\langle T, B\rangle + \tau\langle B, B\rangle - \tau\langle N, N\rangle$$

$$\frac{d}{ds}\langle T,T\rangle = 2k\langle T,N\rangle$$

$$\frac{d}{ds}\langle N,N\rangle = -2k\langle N,T\rangle + 2\tau\langle N,B\rangle$$ by Frenet formulas.

$$\frac{d}{ds}\langle B,B\rangle = -2\tau\langle B,N\rangle$$

We can check that $\langle T,N\rangle = 0$, $\langle T,B\rangle = 0$, $\langle N,B\rangle = 0$

$\langle T,T\rangle = 1$, $\langle N,N\rangle = 1$, $\langle B,B\rangle = 1$

is a solution to the system above w/ initial conditions $0,0,0,1,1,1$.

Then by uniqueness, this is the solution to the system above, and

$\{T(s), N(s), B(s)\}$ stays orthonormal for all $s \in I$.

Now we obtain the curve by integrating wrt. s.

$$\alpha(s) = \int_{s\in I} T(s)\,ds$$

integrating component by component.

By FTC, $\alpha'(s) = T(s)$.

$$\alpha''(s) = T'(s) = k(s)N(s)$$

$k(s)$ is the curvature of α.

$$\frac{\langle \alpha' \times \alpha'', \alpha''' \rangle}{|\alpha' \times \alpha''|^2} \underset{\text{arc-length}}{=} \frac{\langle T \times kN, k'N - k^2T + k\tau B \rangle}{|T \times kN|^2}$$

$$= \frac{k\langle B, k'N - k^2T + k\tau B \rangle}{k^2 |B|^2}$$

$$= \frac{\cancel{k^2}\tau}{\cancel{k^2}} \frac{\cancel{|B|^2}}{\cancel{|B|^2}} = \tau.$$

Therefore, α has curvature given by k, torsion given by τ. //

$$\frac{d}{ds} f = \frac{d}{ds}\left(|T-\overline{T}|^2 + |N-\overline{N}|^2 + |B-\overline{B}|^2\right)$$
$$\leq C\left(\sup_{s\in I} |\kappa|, \sup_{s\in I} |\tau|\right) f.$$

$T(s_0) = \overline{T}(s_0)$

$N(s_0) = \overline{N}(s_0)$

$B(s_0) = \overline{B}(s_0)$

<u>Claim:</u> $\begin{cases} \frac{d}{ds} f \leq Cf. \\ f(s_0) = 0. \end{cases} \Rightarrow f(s) = 0$ for all $s \geq s_0$.

<u>Pf of Claim:</u> WLOG can take $s_0 = 0$. Call $M := \max_{[0,\frac{1}{2C}]} |f|$ which is attained at some $x_0 \in [0, \frac{1}{2C}]$ by continuity of f.

Then by $\frac{d}{ds} f \leq Cf$, we have for any $s \in [0, \frac{1}{2C}]$,
$$|f(s) - \cancel{f(0)}| \leq \int_0^s |f'(t)|\,dt \leq \int_0^s C|f(t)|\,dt \leq CM\cdot s.$$ but since $s \leq \frac{1}{2C}$,

in particular, we get $|f(s)| \leq \frac{M}{2}$ for $s \in [0, \frac{1}{2C}]$.

But since M is attained somewhere, we get $|f(x_0)| = M \leq \frac{M}{2} \Rightarrow M = 0$.

So $f = 0$ on $[0, \frac{1}{2C}]$. Extend to all of $[0, \infty)$ by induction (i.e. show $f = 0$ on $[\frac{n}{2C}, \frac{n+1}{2C}]$ for all n)

//.